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## INFERENCE ABOUT

## THE PARETO-TYPE DISTRIBUTION

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### ABSTRACT

Given a sample from the Pareto-type distribution  $P(X \geq x) = L(x)x^{-b}$ , where  $L(x)$  slowly varies as  $x \rightarrow \infty$ , we estimate the tail index  $b$ . For the special case  $L(x) \rightarrow L \equiv \text{const}$  as  $x \rightarrow \infty$  we construct asymptotically normal estimator of  $L$ .

### INTRODUCTION AND RESULTS

Let  $X, X_1, X_2, \dots$  be a sequence of independent identically distributed random variables with distribution

$$(A) \quad p_x \equiv P(X \geq x) = L(x)x^{-b} \quad (b > 0),$$

where (unknown) function  $L(x)$  slowly varies as  $x \rightarrow \infty$ ; number  $b$  is to be estimated.

Well-known estimators of index  $b$  have been introduced by Hill (1975), Teugels (1985), S. Scorgo, Deheuvels, Mason (1985), M. Scorgo, Horvath, Revesz (1987). Novak and Utev (1990) investigated the estimator

$$t_n = \frac{\sum_{i=1}^n \ln(X_i/N) 1\{X_i \geq N\}}{\sum_{i=1}^n 1\{X_i \geq N\}},$$

where  $N \equiv N(n)$  is chosen so that  $np_N \rightarrow \infty$  as  $n \rightarrow \infty$ . They proved that

$$(1) \quad (np_n)^{1/2} (t_n - 1/b) \rightarrow \mathcal{N}(0; b^{-2}) \quad (n \rightarrow \infty)$$

if and only if  $np_N g^2(N) \rightarrow 0 \quad (n \rightarrow \infty)$ , where

$$g(N) = M\{\ln(X/N) | X \geq N\} - b^{-1}$$

They considered also the situation

$$L(x) = L \cdot (1 + v_x)$$

(B)

$$L \equiv \text{const}, \quad v_x \rightarrow 0 \text{ as } x \rightarrow \infty.$$

In the assumption

$$(2) \quad (\ln N)^2 / np_N \rightarrow 0, \quad g(n)(\ln N) \rightarrow a \in \mathbb{R} \quad (n \rightarrow \infty)$$

they proved that for  $n \rightarrow \infty$  there holds

$$(3) \quad L_n^* \equiv n^{-1} N^{1/t_n} S_n \exp(ab^2) \xrightarrow{p} L,$$

where  $S_n = \sum_{i=1}^n 1\{X_i \geq N\}$ .

The purpose of this article is to generalize results of (1) and (3).

**Theorem 1.** If  $g(N) \rightarrow 0$  as  $n \rightarrow \infty$  then

$$(4) \quad t_n \xrightarrow{p} (1/b) \quad (n \rightarrow \infty)$$

If  $\sqrt{np_N} g(N) \rightarrow A \in \mathbb{R}$  as  $n \rightarrow \infty$  then for  $n \rightarrow \infty$  we have

$$(5) \quad \left\{ (t_n^{-1/b}) S_n^{1/2} - A \right\} \cdot t_n^{-1} \Rightarrow \mathcal{N}(0; 1)$$

**Corollary.** If  $\sqrt{np_N} g(N) \rightarrow A \in \mathbb{R}$  as  $n \rightarrow \infty$  then for  $n \rightarrow \infty$  we have

$$(5') \quad \left\{ t_n^2 (t_n^{-1/b}) S_n^{1/2} + A \right\} \cdot t_n^{-1} \Rightarrow \mathcal{N}(0; 1)$$

Consider now the situation (B).

**Theorem 2.** Let  $\{r_n, n \geq 1\}$  be a sequence of positive numbers such that the following conditions hold for  $n \rightarrow \infty$ :

- (i)  $r_n \rightarrow \infty$
- (ii)  $r_n \sqrt{np_N} \rightarrow B \in \mathbb{R}$
- (iii)  $b^2 r_n (a - g(N) \cdot \ln N) \rightarrow C \in \mathbb{R} \quad (a \in \mathbb{R})$
- (iv)  $(\ln N) (np_N)^{-1/2} r_n \rightarrow c < \infty$
- (v)  $r_n g(N) \rightarrow d \in \mathbb{R}$

Then for  $n \rightarrow \infty$  we have

$$(6) \quad r_n (L_n/L - 1) \Rightarrow \mathcal{N}(B+C+D; b^2 c^2),$$

where  $D = -2ab^3d$ ,  $L_n = n^{-1} N^{1/t_n} S_n \exp(a/t_n^2)$ .

Note that assumption (2) suffices (i)-(v) .

Let us consider special cases

$$(a) \quad v_x = O(e^{-\beta x}) \quad (\beta > 0)$$

$$(b) \quad v_x = O(x^{-\beta}) \quad (\beta > 0)$$

$$(c) \quad v_x = O((\ln x)^{-\beta}) \quad (\beta > 1)$$

In the light of (8) it is easy to see that rate of convergence of  $g(N)$  and  $v_N$  to zero as  $N \rightarrow \infty$  is one and the same. Conditions of theorem 1 will be fulfilled for  $r_n \propto \min\{1/g(N); \sqrt{np_N}\}/(\ln N)$ . Hence, convergence (6) holds with the following rate of  $r_n$ :

$$(a) \quad \{n(\ln n)^{-b}\}^{1/2}/(\ln \ln n) \quad N(n) = \frac{(\ln n - b \ln \ln n)}{2\beta} + O(1)$$

$$(b) \quad n^{\beta/(b+2\beta)}/(\ln n) \quad N(n) \propto n^{1/(b+2\beta)}$$

$$(c) \quad (\ln n)^{\beta-1} \quad N(n) \propto (n(\ln n)^{-2\beta})^{1/b}$$

#### PROOFS

Proof of theorem 1. Let

$$Y_i = 1\{x_i \geq N\}, \quad Z_i = Y_i \ln(X_i/N)$$

There follows from the definition (A) that

$$M \exp(itZ_1) = 1 + \int_0^\infty (e^{itz} - 1) P(Z_1 \in dz)$$

$$\begin{aligned}
 (7) \quad &= 1 + it \int_0^{\infty} e^{ity} P(Z_1 \geq y) dy \\
 &= 1 + it p_N (b - it)^{-1} + it p_N g(N, t),
 \end{aligned}$$

where

$$g(N, t) = \int_0^{\infty} e^{ity} \{L(Ne^y)/L(N) - 1\} e^{-by} dy$$

A remarkable fact is that

$$(8) \quad g(N) = \int_0^{\infty} \{L(Ne^y)/L(N) - 1\} e^{-by} dy$$

Taking into account (8) and properties of slowly varying functions (Seneta, 1976, p.2-6) we derive that for  $N \rightarrow \infty$

$$g(N) \rightarrow 0,$$

$$(9) \quad |g(N) - g(N, t)| = o(|t|)$$

It is easy to check that

$$\begin{aligned}
 (10) \quad &M \exp\{it(S_n - np_N)\} = \\
 &= \exp\{np_N(e^{it} - 1 - it) + O(np_N^2 t^2)\}
 \end{aligned}$$

First assertion of theorem 1 follows from (7), (9), (10).

Let us proof that for  $n \rightarrow \infty$

$$(11) \quad \sum_{j=1}^n [Z_j - Y_j b^{-1} - p_N g(N)] (np_N)^{-1/2} \Rightarrow \mathcal{N}(0; b^{-2})$$

It is easy to see that (11) entails (5).

Similarly to (7) we derive

$$\begin{aligned}
 (12) \quad &M \exp(it(Z - Yb^{-1})) \\
 &= 1 + p_N \left[ -t^2/2b^2 + ite^{-it/b} g(N, t) + O(t^3) \right]
 \end{aligned}$$

The convergence (9) follows from (7) and (12).

Proof of theorem 1 completed.

Remark. It is easy to see that assertions of theorem 1 and corollary remain true if we substitute  $g(N)S_n^{1/2}$  for  $A$  in (5) and (5').

Proof of theorem 2. There follows from the definition (3) that

$$(13) \quad \frac{L_n}{L} = (np_N)^{-1} S_n \exp(at_n^{-2} + (t_n^{-1} - b) \cdot \ln N) \cdot (1 + v_N)$$

Hence, it is sufficient to prove that for  $n \rightarrow \infty$  there holds

$$\tau_n r_n \ln(L_n/L) \Rightarrow \mathcal{N}(B+C+D; b^2 c^2),$$

where  $\tau_n = 1\{S_n \geq np_N/2\}$ . Note that

$$(14) \quad \begin{aligned} & \left| \tau_n \ln(S_n / np_N) - \tau_n \sum_{i=1}^n (Y_i - p_N) / np_N \right| \leq \\ & \leq 2 \left\{ \sum_{i=1}^n (Y_i - p_N) / np_N \right\}^2 \end{aligned}$$

There follows from (10), (13), (14), (i), (ii), (iv) that we need only to argue the convergence

$$r_n (at_n^{-2} + (t_n^{-1} - b) \cdot \ln N) \Rightarrow \mathcal{N}(C+D; b^2 c^2) \quad (n \rightarrow \infty)$$

One derives from (7), (12), (iv)-(v) that for  $n \rightarrow \infty$

$$r_n(t_n^{-1} - b) \equiv$$

$$\equiv - \frac{b^2 r_n \sum_{i=1}^n (Z_i - Y_i b^{-1} - g(N) p_N)}{np_N} \cdot \frac{np_N}{\sum_{i=1}^n b Z_i} \xrightarrow{p} -b^2 d$$

Hence,  $ar_n(t_n^{-2} - b^2) \xrightarrow{p} D$  as  $n \rightarrow \infty$ .

Now our purpose is to show that for  $n \rightarrow \infty$  there holds

$$b^2 r_n \left\{ a - \sum_{i=1}^n (Z_i - Y_i b^{-1}) (\ln N) / \sum_{i=1}^n b Z_i \right\} \rightarrow$$

$$(15) \quad \Rightarrow \mathcal{N}(C; b^2 c^2)$$

Taking into account (7), (12), (iv), (v), one can see that it is sufficient to prove the convergence

$$b^2 r_n \left\{ a - (np_N)^{-1} \sum_{i=1}^n (Z_i - Y_i b^{-1}) (\ln N) \right\} \rightarrow$$

$$(16) \quad \Rightarrow \mathcal{N}(C; b^2 c^2) \quad (n \rightarrow \infty)$$

There follows from (12) and (iv) that

$$b^2 r_n (\ln N) \sum_{i=1}^n (Z_i - Y_i b^{-1} - g(N) p_N) / np_N \rightarrow$$

$$(17) \quad \Rightarrow \mathcal{N}(0; b^2 c^2) \quad (n \rightarrow \infty)$$

The assertion (16) follows from (17) and (iii).

Theorem 2 is proved.



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